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Note

Another simple proof of Graham and Pollak's theorem

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ABSTRACT

Let T be a tree with n vertices, and D_n be the distance matrix of T. Graham and Pollak (1971) discovered an elegant formula for the determinant of D_n: $\det(D_n) = -(n-1)(-2)^{n-2}$. It is surprising that it depends only on the order of T, not on the specific structure of T. By virtue of the classical Dodgson's determinant-evaluation rule, Yan and Yeh (2006) presented a simple proof of the formula above. In this note, we give another simple proof, based on a homogeneous linear three-term recurrence relation: $\det(D_n) + 4\det(D_{n-1}) + 4\det(D_{n-2}) = 0$.

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1. Introduction

Let T be a tree with n vertices whose vertices are denoted by $1, 2, \ldots, n$. Let $D_n = (d_{ij})_{n\times n}$ be the distance matrix of T, where d_{ij} represents the distance from i to j in T.

A remarkable formula for the determinant of D_n was found by Graham and Pollak [7] in 1971:

$$\det(D_n) = -(n-1)(-2)^{n-2},$$

which depends only on the order n of T, independent of the structure of T. More than thirty years later, Yan and Yeh [9] simplified the proof, by exploiting the classical Dodgson's determinant-evaluation rule. More related extensions can be found in [1–6,8,10,11].

It is worth mentioning that a nonhomogeneous linear two-term recurrence relation was deduced in [7,9]:

$$\det(D_n) = -2\det(D_{n-1}) + (-1)^{n-1}2^{n-2}.$$

And the remarkable Dodgson's determinant-evaluation rule actually leads to a quadratic three-term recurrence relation [9]:

$$\det(D_n)\det(D_{n-2}) = (\det(D_{n-1}))^2 - 2^{2n-6}.$$

In this note, based on a simpler homogeneous linear three-term recurrence relation:

$$\det(D_n) + 4\det(D_{n-1}) + 4\det(D_{n-2}) = 0,$$

together with the initial conditions $\det(D_1) = 0$ and $\det(D_2) = -1$, we could give another simple proof for $\det(D_n) = -(n-1)(-2)^{n-2}$.

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2. Proofs

There is an obvious observation: If $n \geq 3$, then there exists a leaf (vertex of degree 1) in T such that its neighbor node is of degree 2 or adjacent to another leaf. For example, assume that v is a terminal vertex of a diametrical path of T, then v is a leaf in T, whose neighbor node is of degree 2 or adjacent to another leaf.

Case 1. There exists a leaf in T whose neighbor node is adjacent to another leaf.

Assume that the vertices $(n-1)$ and n are two leaves of T sharing the common neighbor (the vertex $(n-2)$). Accordingly, D_n is of the form:

$$D_n = \begin{bmatrix} D_{n-3} & \boldsymbol{\alpha} & \boldsymbol{\alpha}+\boldsymbol{e} & \boldsymbol{\alpha}+\boldsymbol{e} \\ \boldsymbol{\alpha}^T & 0 & 1 & 1 \\ \boldsymbol{\alpha}^T+\boldsymbol{e}^T & 1 & 0 & 2 \\ \boldsymbol{\alpha}^T+\boldsymbol{e}^T & 1 & 2 & 0 \end{bmatrix}$$

for some column vector $\boldsymbol{\alpha}$, where $\boldsymbol{e}^T = (1, 1, \ldots, 1)$. Let $\mathcal{R}_i$ ($\mathcal{C}_i$, respectively) be the ith row (column, respectively) of D_n.

In particular,

$$\det \begin{bmatrix} D_{n-3} & \boldsymbol{\alpha} & \boldsymbol{\alpha}+\boldsymbol{e} \\ \boldsymbol{\alpha}^T & 0 & 1 \\ \boldsymbol{\alpha}^T+\boldsymbol{e}^T & 1 & 0 \end{bmatrix}$$

is the determinant of distance matrix of $T - \{n\}$, thus it is actually equal to $\det(D_{n-1})$, and

$$\det \begin{bmatrix} D_{n-3} & \boldsymbol{\alpha} \\ \boldsymbol{\alpha}^T & 0 \end{bmatrix}$$

is the determinant of distance matrix of $T - \{n-1, n\}$, which is $\det(D_{n-2})$. From which, we can deduce that

$$\begin{aligned} & -2\det \begin{bmatrix} D_{n-3} & \boldsymbol{\alpha} & \boldsymbol{\alpha}+\boldsymbol{e} \\ \boldsymbol{\alpha}^T & 0 & 1 \\ \boldsymbol{\alpha}^T+\boldsymbol{e}^T & 1 & 0 \end{bmatrix} - 2\det \begin{bmatrix} D_{n-3} & \boldsymbol{\alpha} & \boldsymbol{\alpha}+\boldsymbol{e} \\ \boldsymbol{\alpha}^T & 0 & 1 \\ \boldsymbol{\alpha}^T+\boldsymbol{e}^T & 1 & 2 \end{bmatrix} \\ = & -2\det \begin{bmatrix} D_{n-3} & \boldsymbol{\alpha} & \boldsymbol{\alpha}+\boldsymbol{e} \\ \boldsymbol{\alpha}^T & 0 & 1 \\ \boldsymbol{\alpha}^T+\boldsymbol{e}^T & 1 & 0 \end{bmatrix} - 2\det \begin{bmatrix} D_{n-3} & \boldsymbol{\alpha} & \boldsymbol{\alpha}+\boldsymbol{e} \\ \boldsymbol{\alpha}^T & 0 & 1 \\ \boldsymbol{\alpha}^T+\boldsymbol{e}^T & 1 & 0 \end{bmatrix} - 2\det \begin{bmatrix} D_{n-3} & \boldsymbol{\alpha} & \boldsymbol{0} \\ \boldsymbol{\alpha}^T & 0 & 0 \\ \boldsymbol{\alpha}^T+\boldsymbol{e}^T & 1 & 2 \end{bmatrix} \\ = & -4\det \begin{bmatrix} D_{n-3} & \boldsymbol{\alpha} & \boldsymbol{\alpha}+\boldsymbol{e} \\ \boldsymbol{\alpha}^T & 0 & 1 \\ \boldsymbol{\alpha}^T+\boldsymbol{e}^T & 1 & 0 \end{bmatrix} - 4\det \begin{bmatrix} D_{n-3} & \boldsymbol{\alpha} \\ \boldsymbol{\alpha}^T & 0 \end{bmatrix} \\ = & -4\det(D_{n-1}) - 4\det(D_{n-2}). \end{aligned} \tag{1}$$

Applying elementary transformations $\mathcal{R}_n - \mathcal{R}_{n-1}$ to D_n, together with (1), we get

$$\begin{aligned} \det(D_n) &= \det \begin{bmatrix} D_{n-3} & \boldsymbol{\alpha} & \boldsymbol{\alpha}+\boldsymbol{e} & \boldsymbol{\alpha}+\boldsymbol{e} \\ \boldsymbol{\alpha}^T & 0 & 1 & 1 \\ \boldsymbol{\alpha}^T+\boldsymbol{e}^T & 1 & 0 & 2 \\ \boldsymbol{0}^T & 0 & 2 & -2 \end{bmatrix} \\ &= -2\det \begin{bmatrix} D_{n-3} & \boldsymbol{\alpha} & \boldsymbol{\alpha}+\boldsymbol{e} \\ \boldsymbol{\alpha}^T & 0 & 1 \\ \boldsymbol{\alpha}^T+\boldsymbol{e}^T & 1 & 0 \end{bmatrix} - 2\det \begin{bmatrix} D_{n-3} & \boldsymbol{\alpha} & \boldsymbol{\alpha}+\boldsymbol{e} \\ \boldsymbol{\alpha}^T & 0 & 1 \\ \boldsymbol{\alpha}^T+\boldsymbol{e}^T & 1 & 2 \end{bmatrix} \\ &= -4\det(D_{n-1}) - 4\det(D_{n-2}). \end{aligned}$$

Case 2. There exists a leaf in T whose neighbor node is of degree 2.

The deduction of this case is very similar to Case 1. We assume that the vertex n is a leaf of T whose unique neighbor (the vertex $n-1$) is of degree 2, and the vertex $n-2$ is the other neighbor of the vertex $n-1$ in T. This time, D_n is of the form:

$$D_n = \begin{bmatrix} D_{n-3} & \boldsymbol{\beta} & \boldsymbol{\beta}+\boldsymbol{e} & \boldsymbol{\beta}+2\boldsymbol{e} \\ \boldsymbol{\beta}^T & 0 & 1 & 2 \\ \boldsymbol{\beta}^T+\boldsymbol{e}^T & 1 & 0 & 1 \\ \boldsymbol{\beta}^T+2\boldsymbol{e}^T & 2 & 1 & 0 \end{bmatrix}$$

for some column vector $\boldsymbol{\beta}$.

Beginning with D_n, applying three elementary transformations successively: $\mathcal{R}_n - 2\mathcal{R}_{n-1}$, $\mathcal{R}_n + \mathcal{R}_{n-2}$, $\mathcal{C}_n + \mathcal{C}_{n-2}$, finally it leads to

$$\begin{aligned}
\det(D_n) &= \det \begin{bmatrix} D_{n-3} & \boldsymbol{\beta} & \boldsymbol{\beta}+\boldsymbol{e} & 2\boldsymbol{\beta}+2\boldsymbol{e} \\ \boldsymbol{\beta}^T & 0 & 1 & 2 \\ \boldsymbol{\beta}^T+\boldsymbol{e}^T & 1 & 0 & 2 \\ \mathbf{0}^T & 0 & 2 & 0 \end{bmatrix} \\
&= -2\det \begin{bmatrix} D_{n-3} & \boldsymbol{\beta} & 2\boldsymbol{\beta}+2\boldsymbol{e} \\ \boldsymbol{\beta}^T & 0 & 2 \\ \boldsymbol{\beta}^T+\boldsymbol{e}^T & 1 & 2 \end{bmatrix} \\
&= -2\det \begin{bmatrix} D_{n-3} & \boldsymbol{\beta} & \boldsymbol{\beta}+\boldsymbol{e} \\ \boldsymbol{\beta}^T & 0 & 1 \\ \boldsymbol{\beta}^T+\boldsymbol{e}^T & 1 & 0 \end{bmatrix} - 2\det \begin{bmatrix} D_{n-3} & \boldsymbol{\beta} & \boldsymbol{\beta}+\boldsymbol{e} \\ \boldsymbol{\beta}^T & 0 & 1 \\ \boldsymbol{\beta}^T+\boldsymbol{e}^T & 1 & 2 \end{bmatrix} \\
&= -4\det(D_{n-1}) - 4\det(D_{n-2}),
\end{aligned}$$

where the last equation follows from (1), just by replacing $\boldsymbol{\alpha}$ by $\boldsymbol{\beta}$.

In conclusion, we always have the three-term recurrence relation:

$$\det(D_n) + 4\det(D_{n-1}) + 4\det(D_{n-2}) = 0.$$

Combining with the initial conditions $\det(D_1) = 0$ and $\det(D_2) = -1$, it is not hard to obtain its solution

$$\det(D_n) = -(n-1)(-2)^{n-2}.$$

CRediT authorship contribution statement

Zhibin Du: Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Software, Writing - original draft . **Jean Yeh:** Data curation, Project administration, Resources, Supervision, Validation, Visualization, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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